J. Fluid Mech. (1975), *vol.* 70, *part* 4, *pp.* 663–671
Printed in Great Britain

Water waves and conjugate streams

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(Received 5 May 1974 and in revised form 3 February 1975)

Steady plane periodic waves on the surface of an ideal liquid above a horizontal bottom are considered. The flow is irrotational. Let Q denote the volume flow rate, R the total head and S the flow force for the wave train. Bounds on wave properties are obtained in terms of the properties of (i) the conjugate streams with the same Q and R, and (ii) the conjugate streams with the same Q and S.

1. Introduction

In appropriate non-dimensional variables, as in Benjamin & Lighthill (1954), the problem for periodic water waves (defined in physical terms in the abstract above) is as follows. Here $z = x+iy$ is the complex position co-ordinate,

$$\chi = \phi + i\psi$$

is the complex potential and $w = (dz/d\chi)^{-1} = u - iv$ is the complex velocity. Define $\Omega \equiv \{\chi = \phi + i\psi \mid -\infty < \phi < \infty, 0 < \psi < 1\}$. Consider the set of functions $z(\chi)$ holomorphic in Ω with both $w(\chi)$ and $z(\chi)$ continuous on $\bar{\Omega}$ such that

$$y = 0 \quad \text{on} \quad \psi = 0 \quad \text{for all } \phi, \tag{1.1a}$$

$$\tfrac{1}{2}q^2 + y = \tfrac{3}{2}R = \text{constant} \quad \text{on} \quad \psi = 1 \quad \text{for all } \phi, \tag{1.1b}$$

where $$q \equiv |w(\phi + i)|,$$

$$y \text{ is even in } \phi, \tag{1.1c}$$

$$y \text{ is periodic in } \phi \quad (\text{with period } \Lambda). \tag{1.1d}$$

Define $$c \equiv \Lambda/\lambda, \quad \lambda \equiv x(\tfrac{1}{2}\Lambda, 0) - x(-\tfrac{1}{2}\Lambda, 0)$$

and $$\eta(\phi) \equiv y(\phi, 1) \quad \text{for all } \phi.$$

The function $z(\chi) = h\chi$, where h is a real constant, satisfies (1.1). Such solutions are called uniform streams. Solutions for which $w(\chi)$ is not constant are called waves. Solutions for which $u > 0$ everywhere and $v \geqslant 0$ for $-\frac{1}{2}\Lambda \leqslant \phi \leqslant 0$, like that shown in figure 1, will be considered. Wave solutions with these properties are known to exist (Krasovskii 1960, 1961).

We define the heights of the wave at the crest and at the trough respectively by

$$h_c \equiv \sup_{-\infty<\phi<\infty} \eta(\phi), \quad h_t \equiv \inf_{-\infty<\phi<\infty} \eta(\phi).$$

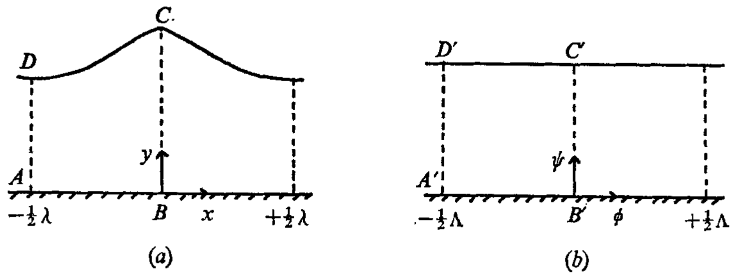


FIGURE 1. (*a*) z plane. (*b*) χ plane.

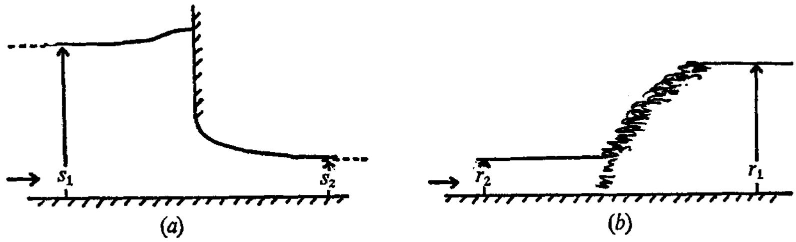


FIGURE 2. (*a*) The sluice gate: a transition where R is constant. (*b*) The bore: a transition where S is constant.

It is known that, for any solution of (1.1),

$$\tfrac{3}{2}S \equiv \tfrac{3}{2}R\eta - \tfrac{1}{2}\eta^2 + \tfrac{1}{2}\int_0^1 u(\phi, \psi)\, d\psi \tag{1.2}$$

is a constant independent of ϕ (Benjamin & Lighthill 1954).

It is shown in propositions $1R$ and $1S$ that, for any wave solution of (1.1), $R > 1$ and $S > 1$. (The inequalities are trivial for uniform streams.) It is a long-standing conjecture (not finally settled) that, for any fixed $R > 1$, there exists a one-parameter family of wave solutions of (1.1) with h_c between a minimum value s_1, defined in equation (2.1), and a maximum value which is less than or equal to $h_s \equiv \frac{3}{2}R$.

2. Conjugate streams

We now state some simple relations for the conjugate streams with which we shall compare the water waves. When R (or S) is fixed the *conjugate streams* are traditionally illustrated as in figure 2(*a*) (or figure 2*b*).

In figure 2(*a*) the conjugate depths s_j satisfy, or more accurately are defined by,

$$s_j^3 - \tfrac{3}{2}Rs_j^2 + \tfrac{1}{2} = 0. \tag{2.1}$$

For $R > 1$ this has only two positive roots s_1 and s_2, with

$$(3R)^{-\frac{1}{2}} < s_2 < R^{-\frac{1}{2}} < 1 < R < s_1 < \tfrac{3}{2}R.$$

As j takes the values 1 and 2 let j' take the values 2 and 1. Then s_1 and s_2 are related by

$$s_{j'} = [1 + (1 + 8s_j^3)^{\frac{1}{2}}]/4s_j^2,$$

and so $$\tfrac{1}{2}(s_1+s_2) = s_1^2 s_2^2 > 1.$$

Similarly, in figure 2(*b*), the conjugate depths r_j are defined by

$$\tfrac{1}{2}r_j^3 - \tfrac{3}{2}Sr_j + 1 = 0. \tag{2.2}$$

For $S > 1$ this has only two positive roots r_1 and r_2, with

$$2/(3S) < r_2 < S^{-1} < 1 < S^{\frac{1}{2}} < r_1 < (3S)^{\frac{1}{2}}.$$

With the same notation as above, r_1 and r_2 are related by

$$r_{j'} = \tfrac{1}{2}[-r_j^2 + (r_j^4 + 8r_j)^{\frac{1}{2}}],$$

and so $$2/(r_1+r_2) = r_1 r_2 < 1.$$

3. Bounds for water waves

Inequalities (3.1)–(3.3) below are basic to everything that follows.

Maximum principle result. Consider wave solutions of (1.1). Let q_t and q_c denote the flow speeds on $\psi = 1$ at the trough D and crest C respectively. Then

$$q_c < |w(\chi)| < q_t \quad \text{for all } \chi \text{ in } \Omega.$$

In particular, $$q_c h_c < 1 < q_t h_t. \tag{3.1}$$

Further $$q_t > \int_0^1 u(\phi,\psi)\,d\psi \geqslant \int_0^1 u(0,\psi)\,d\psi > h_c^{-1} > q_c \tag{3.2}$$

and $$\int_0^1 u(\pm\tfrac{1}{2}\Lambda,\psi)\,d\psi > h_t^{-1}. \tag{3.3}$$

Proof. Since $w(\chi)$ is holomorphic in Ω the maximum value of q^2 occurs on the boundary. Further, $w(\chi)^{-1}$ is holomorphic in Ω because $w(\chi)$ has no zeros in Ω. (Since u is harmonic and non-constant it cannot have a minimum in the interior Ω. The velocity u is also non-negative, thus it has no zero in Ω and so the result follows.) Thus the minimum value of q^2 occurs on the boundary.

From our assumption that the free surface rises monotonically between the trough D and crest C, using the Bernoulli equation we have

$$q_c < |w(\phi+i)| < q_t.$$

Again from the geometry of the flow sketched in figure 1, using the Cauchy–Riemann equations $u_\phi = -v_\psi$ and $u_\psi = +v_\phi$, u decreases from D to A (where $u_\psi = v_\phi \geqslant 0$), from A to B (where $u_\phi = -v_\psi \leqslant 0$) and from B to C (where $u_\psi = v_\phi \leqslant 0$). This establishes the inequality $q_c < |w(\chi)| < q_t$ for all χ in Ω. Then (3.1) follows by using this, at $\phi = 0$ and $\phi = -\tfrac{1}{2}\Lambda$, in

$$\eta(\phi) = \int_0^1 \frac{u(\phi,\psi)}{|w(\phi+i\psi)|^2}\,d\psi.$$

To prove (3.2) and (3.3) the Schwarz inequality is applied, at $\phi = 0$ and $\phi = -\tfrac{1}{2}\Lambda$ respectively, as follows:

$$1 = \left(\int_0^1 u^{\frac{1}{2}}\frac{1}{u^{\frac{1}{2}}}\,d\psi\right)^2 < \left(\int_0^1 \frac{d\psi}{u}\right)\left(\int_0^1 u\,d\psi\right).$$

The full statement of (3.2) then follows from

$$\frac{\partial}{\partial \phi}\int_0^1 u(\phi,\psi)\,d\psi = -v(\phi,1) \leqslant 0 \quad \text{for} \quad -\tfrac{1}{2}\Lambda \leqslant \phi \leqslant 0.$$

We remark that (3.1) and with a little more argument (3.2) and (3.3), and also

$$h_c^{-1} \leqslant u(\phi,0) \leqslant h_t^{-1},$$

follow from the Lavrentiev–Serrin comparison theorems (Lavrentiev 1964, p. 19; Serrin 1952*a*, *b*). One 'compares' the wave solution with uniform flows of depths h_c and h_t, and the same volume flow rates.

PROPOSITION 1R. Consider a periodic wave train as before with total head R. Then $R > 1$. Let h_c and h_t be the heights of the crest and trough of the wave. Let s_1 and s_2 denote the depths of subcritical and supercritical uniform streams with the given R. Then

$$s_2 < h_t < s_1 < h_c. \tag{3.4}$$

Proof. Define

$$\mathscr{R}(h) = \tfrac{1}{3}h^{-2} + \tfrac{2}{3}h. \tag{3.5}$$

A graph of $\mathscr{R}(h)$ is given in figure 3(*a*). Note that

$$\min_{h>0} \mathscr{R}(h) = 1.$$

We shall show that

$$\mathscr{R}(h_c) > R > \mathscr{R}(h_t), \tag{3.6}$$

and since $\mathscr{R}(h_t) > 1$ then $R > 1$. Thus s_1 and s_2 are defined as in §2, so that

$$\mathscr{R}(h_c) > \mathscr{R}(s_1) = R = \mathscr{R}(s_2) > \mathscr{R}(h_t).$$

The immediate implications of this are first that $s_2 < h_t < s_1$ and second, since $h_c > h_t$, that $h_c > s_1$. See figure 3(*a*).

Bernoulli gives $\tfrac{1}{2}q_c^2 + h_c = \tfrac{3}{2}R = \tfrac{1}{2}q_t^2 + h_t$. Using (3.1)

$$\mathscr{R}(h_c) = \tfrac{1}{3}h_c^{-2} + \tfrac{2}{3}h_c > \tfrac{1}{3}q_c^2 + \tfrac{2}{3}h_c = R = \tfrac{1}{3}q_t^2 + \tfrac{2}{3}h_t > \tfrac{1}{3}h_t^{-2} + \tfrac{2}{3}h_t = \mathscr{R}(h_t).$$

This establishes (3.6) and hence (3.4).

COROLLARY 1R. *The following inequalities hold*:

$$q_c < h_c^{-1} < s_1^{-1} < h_t^{-1} < q_t < s_2^{-1} < s_1 < h_c < q_c^{-1},$$
$$q_c < h_c^{-1} < s_1^{-1} < s_2 < 1 < s_2^{-1} < s_1 < h_c < q_c^{-1},$$
$$q_c < h_c^{-1} < s_1^{-1} < s_2 < q_t^{-1} < h_t < s_1 < h_c < q_c^{-1}.$$

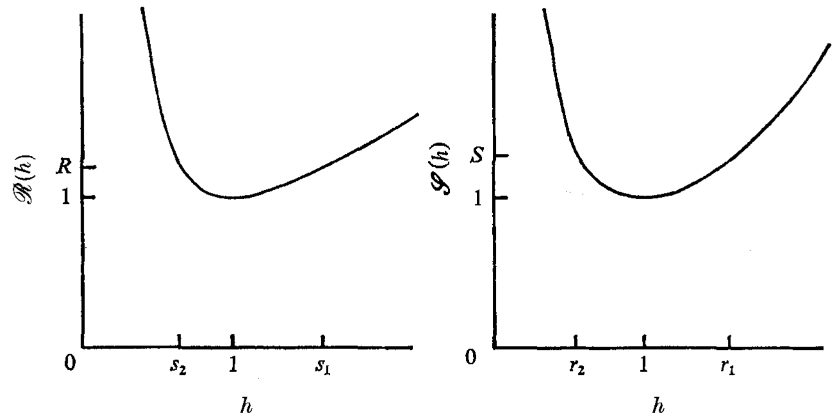
Proof. Since $\tfrac{3}{2}R = \tfrac{1}{2}q^2 + \eta$ inequality (3.4) implies that

$$q_c < s_1^{-1} < q_t < s_2^{-1}.$$

This with (3.4), (2.1) and (3.1) gives the required inequalities.

PROPOSITION 1S. Consider a periodic wave train with flow force S. Then $S > 1$. Let h_c and h_t be as before. Let r_1 and r_2 denote the heights of the subcritical and supercritical uniform streams with the given S. Then

$$r_2 < h_t < r_1. \tag{3.7}$$



(a) FIGURE 3 (b)

Proof. Define
$$\mathscr{S}(h) \equiv \tfrac{2}{3}h^{-1} + \tfrac{1}{3}h^2. \tag{3.8}$$
A graph of $\mathscr{S}(h)$ is given in figure 3(b). Note that
$$\min_{h>0} \mathscr{S}(h) = 1.$$
We shall show that
$$S > \mathscr{S}(h_t), \tag{3.9}$$
and since $\mathscr{S}(h_t) > 1$ then $S > 1$. Thus r_1 and r_2 are defined as in §2, so that
$$\mathscr{S}(r_1) = S = \mathscr{S}(r_2) > \mathscr{S}(h_t),$$
which implies that $r_2 < h_t < r_1$. See figure 3(b).

The flow force [from (1.1b) and (1.2)] is defined by
$$\tfrac{3}{2}S = \tfrac{1}{2}q^2\eta + \tfrac{1}{2}\eta^2 + \tfrac{1}{2}\int_0^1 u(\phi, \psi)\, d\psi.$$
Thus, using inequalities (3.1) and (3.3),
$$\tfrac{3}{2}S > \tfrac{1}{2}h_t^{-1} + \tfrac{1}{2}h_t^2 + \tfrac{1}{2}h_t^{-1} = \tfrac{3}{2}\mathscr{S}(h_t).$$
This establishes (3.9), that $S > \mathscr{S}(h_t)$ and hence (3.7).

COROLLARY 1*S*. The following inequalities hold:
$$r_2 < r_1^{-1} < h_t^{-1} < q_t < r_2^{-1},$$
$$r_2 < r_1^{-1} < 1 < r_1 < r_2^{-1},$$
$$r_2 < q_t^{-1} < h_t < r_1 < r_2^{-1}.$$

Proof. Let $u_t(\psi) = u(-\tfrac{1}{2}\Lambda, \psi)$. Then
$$\tfrac{3}{2}S = \tfrac{1}{2}h_t^2 + \tfrac{1}{2}\int_0^1 \left(\frac{q_t^2}{u_t(\psi)} + u_t(\psi)\right) d\psi.$$
Since $q_t^2/u_t + u_t \geqslant 2q_t$ and since $h_t^2 \geqslant q_t^{-2}$ from (3.1),
$$\tfrac{3}{2}S \geqslant \tfrac{3}{2}\mathscr{R}(q_t) = \tfrac{3}{2}\mathscr{S}(q_t^{-1}).$$

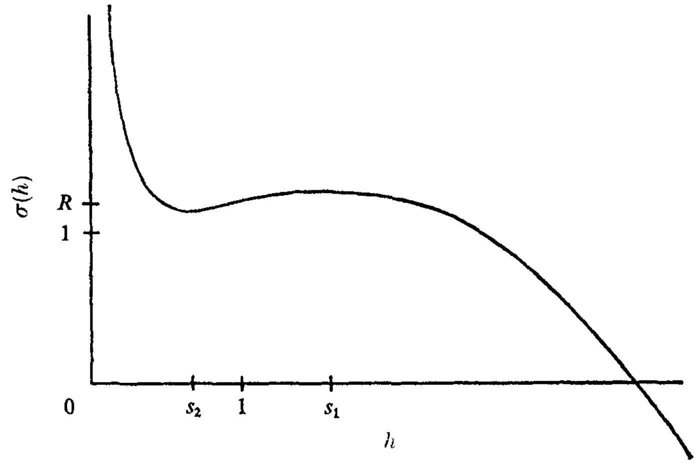


FIGURE 4

Repeating the argument of the previous propositions we obtain

$$r_1^{-1} < q_t < r_2^{-1}. \tag{3.10}$$

This with (3.7), (2.2) and (3.1) gives the required inequalities.

CONJECTURE 1*S*. It is conjectured that

$$r_1 < h_c.$$

This is true if and only if $\mathscr{S}(h_c) > S$.

CONJECTURE 2*S*. It is conjectured that

$$q_c < r_1^{-1}.$$

This is true if and only if $\mathscr{S}(q_c^{-1}) > S$.

Using $q_c < h_c^{-1}$ from (3.1), the truth of conjecture 2*S* follows if the, apparently stronger, conjecture 1*S* is true. Also the truth of conjecture 1*S* follows if the, apparently stronger, conjecture 2, given below, is true.

PROPOSITION 2. Consider a periodic wave train with total head *R* as in proposition 1*R*. Define the flow force *S* using (1.1):

$$\tfrac{3}{2}S = \tfrac{3}{2}R\eta - \tfrac{1}{2}\eta^2 + \tfrac{1}{2}\int_0^1 u(\phi, \psi)\, d\psi.$$

Define $$\tfrac{3}{2}S_j = \tfrac{3}{2}Rs_j - \tfrac{1}{2}s_j^2 + \tfrac{1}{2}s_j^{-1}. \tag{3.11}$$

Then $$S > S_2. \tag{3.12}$$

Proof. Define $$\tfrac{3}{2}\sigma(h) = \tfrac{3}{2}Rh - \tfrac{1}{2}h^2 + \tfrac{1}{2}h^{-1}. \tag{3.13}$$

Then by definition $\sigma(s_j) = S_j$ and $\sigma'(s_j) = 0$. A graph of $\sigma(h)$ is shown in figure 4. We shall show that

$$S \geqslant \sigma(h_t), \tag{3.14}$$

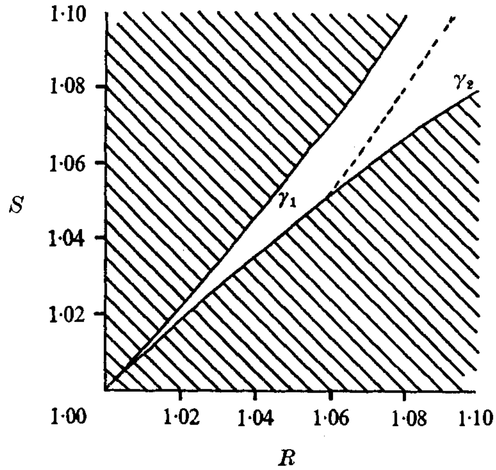


FIGURE 5

and since $s_2 < h_t < s_1$ we then have from the properties of σ above

$$\sigma(h_t) > \sigma(s_2) = S_2.$$

Thus $S > S_2$ as required.

Inequality (3.14) follows immediately from definitions and (3.3) with the minus sign. This completes the proof of (3.14) and hence (3.12).

Similarly, or from the above, for a periodic wave train with total head R, $R < R_2 = \mathscr{R}(s_2)$.

Define $$C(\eta) \equiv \eta^3 - 3R\eta^2 + 3S\eta - 1.$$

Solutions of (1.1) satisfy

$$C(\eta) = \eta \int_0^1 u(\phi, \psi)\, d\psi - 1.$$

All uniform streams have $C(\eta) = 0$. Consider next the roots of the cubic $C(\eta) = 0$. Benjamin & Lighthill (1954) conjecture that this equation must have three real roots whenever R and S correspond to solutions of (1.1). Define

$$\Delta \equiv 3R^2S^2 + 6RS - 1 - 4(R^3 + S^3).$$

All uniform streams have $\Delta = 0$. Benjamin & Lighthill's conjecture is that for values of R and S corresponding to wave solutions of (1.1) $\Delta \geqslant 0$, the condition for three real roots. In the R, S plane sketched in figure 5, γ_1 is the curve $\Delta = 0$ with $R < S$ and γ_2 is the curve $\Delta = 0$ with $R > S$. The unhatched region corresponds to all points with $\Delta > 0$, the hatched region to all points with $\Delta < 0$.

CONJECTURE 2. All wave solutions of (1.1) have values of R and S such that $\Delta > 0$.

We note that if conjecture 2 is true so is conjecture 1S. First, $\Delta > 0$ implies that

$$s_1 > r_1 > 1 > s_2 > r_2.$$

Proposition 1R gives

$$h_c > s_1,$$

so that $h_c > r_1$, which was the content of conjecture 1S.

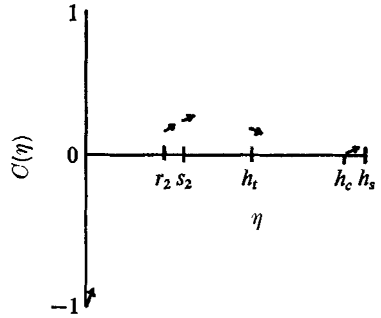


FIGURE 6

Of course, we already have some information on the values of R and S allowed for wave solutions. Propositions 1 imply that $R > 1$ and $S > 1$. Proposition 2 implies that any points (R, S) must lie above and to the left of the line γ_2.

Since $\Delta = -4\{(S-R^2)^3+(-\frac{3}{2}RS+\frac{1}{2}+R^3)^2\}$, a necessary condition for $\Delta > 0$ is the following.

PROPOSITION 3. For any solution of (1.1), $S < R^2$.

Proof.
$$9(R^2-S) = (q^2-\eta)^2+3\eta q^2-3\int_0^1 u(\phi,\psi)\,d\psi.$$

Since
$$\int_0^1 u(-\tfrac{1}{2}\Lambda,\psi)\,d\psi < q_t < q_t^2 h_t$$

the inequality is satisfied at the trough, and hence everywhere.

A final inequality, generalizing an inequality on solitary waves found by Starr (1947) and reported in Long (1956) and Keady & Pritchard (1974), follows from $h_c \leqslant h_s = \frac{3}{2}R$. It has long been known that there is a curve within the region $\Delta > 0$ corresponding to waves of greatest height, for which $h_c = h_s$.

PROPOSITION 4. $C(h_s) > 0$, that is, $-\frac{27}{8}R^3+\frac{9}{2}RS-1 > 0$.

Proof. This follows from the requirement that $h_c \leqslant h_s$ and the following facts concerning $C(\eta)$ and $C'(\eta)$;

$$C'(\eta) = 3\eta^2-6R\eta+3S = -q^2\eta+\int_0^1 u\,d\psi \quad \text{for} \quad h_t \leqslant \eta \leqslant h_c.$$

We have $\quad C(0) < 0, \quad C(r_2) > 0, \quad C(s_2) > 0, \quad C(h_t) > 0, \quad C(h_c) > 0$

and $\quad C'(0) > 0, \quad C'(r_2) > 0, \quad C'(s_2) > 0, \quad C'(h_t) < 0, \quad C'(h_c) > 0.$

From the form of the cubic function $C(\eta)$ it is evident (see figure 6) that $C(h_s) > 0$, and this completes the proof. The curve $C(\frac{3}{2}R) = 0$ is indicated by the dashed line in figure 5.

The authors thank Professor T. B. Benjamin for suggesting the problem while one of us (G. K.) was working at the Fluid Mechanics Research Institute at the University of Essex.

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